Beta Matrix and Common Factors in Stock Returns

Seung C. Ahn, Alex R. Horenstein, and Na Wang*

We consider the estimation methods for the rank of a beta matrix corresponding to a multifactor model and study which method would be appropriate for data with a large number of assets. Our simulation results indicate that a restricted version of Cragg and Donald's (1997) Bayesian Information Criterion estimator is quite reliable for such data. We use this estimator to analyze some selected asset pricing models with U.S. stock returns. Our results indicate that the beta matrix from many models fails to have full column rank, suggesting that risk premiums in these models are under-identified.

Key Words: factor models, beta matrix, rank, risk factors, and asset prices.

JEL Classification: C01, C23, C31, G12

----

*Ahn (corresponding author), miniahn@asu.edu, W. P. Carey School of Business, Arizona State University, and Sogang University; Horenstein, horenstein@bus.miami.edu, School of Business, University of Miami; and Wang, na.wang@hofstra.edu, Zarb School of Business, Hofstra University.

We are grateful to Raymond Kan, George Korniotis, Alok Kumar, Richard Roll, Aurelio Vazquez, Todd Pronno, the participants at Columbia University, Instituto Tecnológico Autónomo de México, Universidad Autónoma de Nuevo León, Sogang University, Hofstra University, Arizona State University, University of Miami, Hitotsubashi University, and numerous conferences at which we presented this paper. We also thank Laura Xiaolei Liu, Jonathan Lewellen, and Stefan Nagel for sharing their data with us.

I. Introduction

Treynor (1962), Sharpe (1964), Lintner (1965), and Mossin (1966) developed the Capital Asset Pricing Model (CAPM), which laid out the foundations of modern asset pricing theory. In later work, Merton's (1972) intertemporal CAPM and Ross's (1976) Arbitrage Pricing Theory (APT) suggested that investors may consider multiple risk sources when making their investment decisions, providing the framework for multifactor asset pricing models. Since the advent of these 2 theories, many variables have been proposed as proxies for the common factors that drive co-movement in asset returns. We refer to these variables as "empirical factors." Two well-known examples, among many, are the 3 factors of Fama and French (1993) and the 5 macroeconomic factors of Chen, Roll, and Ross (1986).[1]

Multifactor models typically predict that the expected return on a risky asset is determined by its quantity of undiversifiable risks (betas) and the necessary rewards (premiums) to induce investors to bear the risks. Under an asset pricing model with K empirical factors, the K betas of an asset are simply the coefficients of regressors in a regression model in which the dependent variable is the asset's (excess) return and the regressors are the K factors. The $N \times K$ matrix of betas is the matrix of K betas from each of the N different asset returns. In order to examine the empirical relevance of an asset pricing model, researchers need to estimate the beta matrix and the risk premiums related to each of the K factors. However, the risk premiums are identifiable and can be consistently estimated only if the $N \times K$ *true* (but unobservable) beta matrix has full column rank; that is, all of the columns in the true beta matrix need to be linearly

[1] Harvey, Liu, and Zhu (2015) categorize 314 empirical factors from 311 different papers published in top-tier finance journals and current working papers since 1967.

independent.[2] Thus, it is important to estimate the rank of the true beta matrix before estimating risk premiums.

The rank of the beta matrix also provides important inferences on the number of common factors in risky asset returns. It increases whenever a factor having an identifiable risk premium or price (either zero or non-zero) is added to an asset pricing model. Thus, the rank of the beta matrix corresponding to a set of factors equals the number of the factors whose prices are identifiable. Such factors are important determinants of co-movement in individual returns, regardless of whether their prices are zeros or non-zeros.[3]

Many methods are available to estimate the rank of a matrix. Examples are the methods developed by Anderson (1951), Cragg and Donald (1997), Robin and Smith (2000), and Kleibergen and Paap (2006).[4] These methods are useful for the cases in which both N (the number of assets analyzed) and K (the number of common factors) are sufficiently small compared to the sample size (T, the number of time series observations of asset returns and common factors). For example, Burnside (2016) examines the statistical test methods of Cragg and Donald (1997) and Kleibergen and Paap (2006) and finds that the test methods have good power to detect rank deficient beta matrices for data with $N = 25$, $T = 256$ and $K \leq 4$. He uses

[2] The literature has reported some special cases in which beta matrices fail to have full column rank. Kan and Zhang (1999a), (1999b) consider a case in which all betas corresponding to an empirical factor equal 0. Ahn, Perez, and Gadarowski (2013) find that the estimated market betas have very small cross-sectional variations and that some betas of the 3 factors of Fama and French (1993) are highly multicollinear.

[3] It is worth emphasizing that we are not trying to identify factors with non-zero risk premiums. When a set of factors has a full column rank beta matrix, their prices are identifiable. The prices could be zeros or non-zeros.

[4] Al-Sadoon (2015) provides an excellent survey of the available rank estimation methods.

these methods to check what factor models generate rank-deficient beta matrices and documents that beta matrices from many models fail to have full column rank.

The main purpose of this paper is to find the methods that can accurately estimate the rank of a beta matrix even if *N* and/or *K* are large. A recent study by Lewellen, Nagel, and Shanken (2010) suggests that the relevance of an asset pricing model can be better tested by analyzing a large number of asset returns. Their findings raise 2 questions: (i) Are the available rank estimators accurate even for data with large *N* (e.g., $N \geq 25$)? (ii) Are the estimators more accurate when data with larger *N* are used (e.g., $N \geq 50$)? This paper attempts to answer these questions.

Our estimator of interest is a version of the Bayesian information criterion (BIC) estimator of Cragg and Donald (1997) that is computed under the assumption that return data are independent and identically distributed (i.i.d.) conditionally on empirical factors. We refer to this estimator as the "restricted BIC" (RBIC) estimator. The main findings from our asymptotic analysis and simulations are the following. First, the RBIC estimator is consistent (asymptotically unbiased) even if data are, in fact, conditionally heteroskedastic and/or autocorrelated. Second, the RBIC estimator performs well with data comparable to monthly data of 20 or more years ($T \geq 240$) and when the number of assets analyzed (*N*) is not greater than half of the number of time series observations ($N \leq T/2$).

Many different multifactor models have been proposed in the literature. Among them, we consider 26 factors from twelve different models: the short-term and long-term reversal factors and the factors from the models of Chen et al. (1986), Fama and French (1993), Jagannathan and Wang (1996), Carhart (1997), Fama and French (2015), Hou, Xue, and Zhang (2015), Lettau and Ludvigson (2001), Lustig and Van Nieuwerburgh (2005), Li, Vassalou, and Xing (2006), Yogo (2006), Santos and Veronesi (2006), and the consumption CAPM (CCAPM). We examine which of these factors have identifiable risk premiums by applying the RBIC method to monthly and quarterly returns on U.S. stock portfolios and individual stocks over the period from 1972 to 2011.

The 2 main results from our actual data analysis are the following. First, many models fail to produce full rank beta matrices. This result is consistent with the findings of Burnside (2016) and Kleibergen and Paap (2006). Second, when all the 26 factors are used in a regression, the rank of the beta matrix is, at most, 5. That is, only a few of the 26 factors appear to have identifiable risk premiums. We use a sequential method to single out such factors.

The rest of this paper is organized as follows. Section II introduces the factor model that we investigate, the BIC estimator, and properties of the RBIC estimator. Section III reports our Monte Carlo simulation results, and Section IV discusses the results from actual data analysis. Some concluding remarks follow in Section V. All of the proofs of our theoretical results are given in the Appendix.

II. Rank Estimation

We begin with the approximate factor model considered in Chamberlain and Rothschild (1983) and Bai and Ng (2002). Let x_{it} be the response variable for the *i*th cross-section unit at time *t* (*i* = 1, 2, ... , *N*, and *t* = 1, 2, ... , *T*). Explicitly, x_{it} can be the (excess) return on asset i at time t. The response variables x_{it} depend on the *K* empirical factors $f_t = (f_{it}, ..., f_{Kt})'$. That is,

(1) $$x_t = \alpha + \mathrm{B} f_t + \varepsilon_t ,$$

where $x_t = (x_{it}, ..., x_{Nt})'$; α is the *N*-vector of individual intercepts; B is the $N \times K$ matrix of factor loadings (beta matrix); and ε_t is the *N*-vector of idiosyncratic components of individual returns at time *t*. The entries in ε_t can be cross-sectionally correlated. We denote the *i*th row of α and B by α_i and $\beta_i = (\beta_{1i}, ..., \beta_{Ki})'$, respectively.

The focus of this paper is the estimation of the rank of the beta matrix B, rank(B). In order to facilitate our discussions about rank estimators, we introduce some notation. Let

$$\hat{\Sigma}_{xf} = T^{-1}\Sigma_{t=1}^{T}(x_t - \bar{x})(f_t - \bar{f})'; \hat{\Sigma}_{ff} = T^{-1}\Sigma_{t=1}^{T}(f_t - \bar{f})(f_t - \bar{f})',$$

where $\bar{f} = T^{-1}\Sigma_{t=1}^{T} f_t$ and x is similarly defined. With this notation, the ordinary least squares (OLS) estimator of B is given by $\hat{\mathrm{B}} = (\hat{\beta}_1, ..., \hat{\beta}_N)' = \hat{\Sigma}_{xf}\hat{\Sigma}_{ff}^{-1}$. Under suitable conditions detailed in the appendix and the assumption of fixed *N*, we can show that as $T \to \infty$,

(2) $$\sqrt{T}\mathrm{vec}(\hat{\mathrm{B}}' - \mathrm{B}') \to_d N(0_{N\times 1}, \Omega),$$

where $\mathrm{vec}(\bullet)$ is a matrix operator stacking all the columns in a matrix into a column vector; Ω is a finite positive definite matrix; and "$\to_d$" means "converges in distribution."

Let $\hat{\Omega}$ be a consistent estimator of Ω, and let A_p and M_p be $K \times p$ and $N \times p$ ($0 \le p \le K$) matrices of full column rank, respectively. Finally, let $\hat{A}_p$ and $\hat{M}_p$ be the minimizers of the objective function

(3) $$\Pi(A_p, M_p, \hat{\Omega}) = T\mathrm{vec}(\hat{\mathrm{B}} - A_p M_p')'\hat{\Omega}^{-1}\mathrm{vec}(\hat{\mathrm{B}} - A_p M_p').$$

Cragg and Donald (1997) show the following:

(4) $$\Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega}) \to_p \infty, \text{ for } p < r,$$

(5) $$\Pi(\hat{A}_r, \hat{M}_r, \hat{\Omega}) \to_p \chi^2_{(N-r)(K-r)}, \text{ for } p = r,$$

where $r = \mathrm{rank}(\mathrm{B})$ and "$\to_p$" means "converges in probability." Based on these findings, they develop 2 different rank estimation methods. One estimator, which they refer to as the Testing Criterion (TC) estimator, is obtained by repeatedly testing the null hypotheses of $r = p$ (*p* = 0, 1, 2, ..., $K - 1$) against the alternative hypothesis of full column rank. Each hypothesis is tested by using $\Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega})$ as a $\chi^2_{(N-p)(K-p)}$ statistic. The TC estimator is the minimum value of *p* that does not reject the hypothesis of $r = p$. If all of the null hypotheses are rejected, the TC estimator equals *K*.

The other estimator, which Cragg and Donald (1997) refer to as the Bayesian information criterion (BIC) estimator, is obtained by finding the value of *p* that minimizes the criterion function

(6) $$C(p) = \Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega}) - \ln(T) \times (N-p)(K-p),$$

where $p = 0, 1, \ldots, K$.

While the TC and BIC estimators have desirable large-sample properties, they are computationally burdensome to use, especially for the cases with large *N*. This is so because A_p and M_p contain a large number of unknown parameters to be estimated, especially for the cases with large *N* and *p*. In unreported experiments, we attempted to compute the TC and BIC estimators using the same simulated data that we use for the results reported in Section III. We observed that standard minimization algorithms failed to find $\hat{A}_p$ and $\hat{M}_p$ too often.

This computational problem can be resolved if we impose some restrictions on Ω. For example, suppose that the error vectors ε_t are i.i.d conditionally on the empirical factors f_t with the conditional variance-covariance matrix $\Sigma_{\varepsilon\varepsilon} = \text{var}(\varepsilon_t \mid f_t)$. The individual errors in ε_t are still allowed to be cross-sectionally correlated; that is, the off-diagonal elements of $\Sigma_{\varepsilon\varepsilon}$ need not be 0. For this case, the computation procedures for the TC and BIC estimators are considerably simplified. When the ε_t are i.i.d. over time, $\hat{\Omega}_R = \hat{\Sigma}_{\varepsilon\varepsilon} \otimes \hat{\Sigma}_{ff}^{-1}$ is a consistent estimator of Ω, where "$\otimes$" means the Kronecker product and $\hat{\Sigma}_{\varepsilon\varepsilon}$ is a consistent estimator of $\Sigma_{\varepsilon\varepsilon}$; *i.e.*,

$$\hat{\Sigma}_{\varepsilon\varepsilon} = (T-K)^{-1}\Sigma_{t=1}^{T}[(x_t - \bar{x}) - \hat{\mathrm{B}}(f_t - \bar{f})][(x_t - \bar{x}) - \hat{\mathrm{B}}(f_t - \bar{f})]'.$$

Cragg and Donald (1997) show that when $\hat{\Omega}_R$ is used for $\hat{\Omega}$,

(7) $$\Pi(\hat{A}_K, \hat{M}_K, \hat{\Omega}_R) = 0;\ \Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega}_R) = T \times \Sigma_{j=1}^{K-p}\hat{\psi}_j,$$

where $p = 0, 1, \ldots, K-1$, $\hat{\psi}_j = \psi_j\left(\hat{\Sigma}_{ff}\hat{\mathrm{B}}'\hat{\Sigma}_{\varepsilon\varepsilon}^{-1}\hat{\mathrm{B}}\right)$, and $\psi_j(\bullet)$ denotes the *j*th smallest eigenvalue of the matrix in the parenthesis. We refer to the TC estimator using (7) as the "restricted TC" (RTC) estimator.

Many previous studies have considered the statistic $\Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega}_R)$. For example, if we replace $\hat{\psi}_j$ by $\ln(1+\hat{\psi}_j)$, $\Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega}_R)$ becomes Anderson's (1951) likelihood ratio (LR) rank test statistic. When $p = r$ and the ε_t are i.i.d. over time, $\Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega}_R)$ and the LR statistic are asymptotically identical. Robin and Smith (2000) show that $\Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega}_R)$ and the LR statistic with $p = r$ are asymptotically a weighted sum of independent χ_1^2 random variables when the ε_t are conditionally heteroskedastic and/or autocorrelated. A difficulty in using $\Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega}_R)$ for the TC estimation is that its asymptotic distribution needs to be simulated using some parameter estimates. As a treatment for this problem, Kleibergen and Paap (2006) propose an alternative statistic that is asymptotically $\chi^2_{(N-p)(K-p)}$ distributed when $p = r$ even if the ε_t are not i.i.d. over time. Their statistic is also asymptotically identical to $\Pi(\hat{A}_p, \hat{M}_p, \hat{\Omega}_R)$ when $p = r$ and the ε_t are i.i.d. over time.

The BIC estimator computed with $\hat{\Omega}_R$ is another consistent estimator of $\mathrm{rank}(\mathrm{B})$ even if the ε_t are conditionally heteroskedastic and/or autocorrelated. We refer to this BIC estimator as the "restricted BIC" (RBIC) estimator, which is the minimizer of the criterion function (6) with (7):

$$C_R(p) = T \times \Sigma_{j=1}^{K-p} \hat{\psi}_j - \ln(T) \times (N-p)(K-p) .$$

While a formal proof is given in the Appendix, a simple explanation for the consistency of the RBIC estimator follows. While Cragg and Donald (1997) have shown the consistency of the BIC estimator using the results (4) and (5), the same consistency result is obtained with $\hat{\Omega}$ in equation (3) replaced by any (asymptotically) positive definite matrix as long as expression (4) holds and $\Pi(\hat{A}_r, \hat{M}_r, \hat{\Omega})$ is $O_p(1)$ (an asymptotically bounded random variable). The statistic $\Pi(\hat{A}_r, \hat{M}_r, \hat{\Omega})$ needs not be an asymptotically χ^2 random variable. Fortunately, Robin and Smith's (2000) results indicate that $\Pi(\hat{A}_r, \hat{M}_r, \hat{\Omega}_R)$ is $O_p(1)$ even if the ε_t are not i.i.d. over time. Thus, the RBIC estimator is consistent under much more general conditions.

Using the RBIC method, we can also test whether or not some betas are cross-sectionally constant by comparing the ranks of 2 matrices: the beta matrix B and its demeaned version, $Q_N \mathrm{B} = (\dot{\beta}_1, ..., \dot{\beta}_N)$, where $Q_N = I_N - N^{-1} 1_N 1_N'$, 1_N is an *N*-vector of ones, $\dot{\beta}_i = \beta_i - \overline{\beta}$ and $\overline{\beta} = N^{-1} \Sigma_{i=1}^N \beta_i$. If a column of B (or a linear combination of the columns of B) is proportional to the *N*-vector of ones, the corresponding column of $Q_N \mathrm{B}$ (or the corresponding linear combination of the columns of $Q_N \mathrm{B}$) becomes a zero vector. Thus, $\mathrm{rank}(Q_N \mathrm{B}) = r - 1$. If no column of B and no linear combination of columns of B are proportional to a vector of ones, B and $Q_N \mathrm{B}$ must have the same ranks. Therefore, comparing the estimated ranks of B and $Q_N \mathrm{B}$, we can determine whether a constant-beta factor exists in f_t.

The RBIC estimator can be easily modified to estimate the rank of the demeaned beta matrix. We define the following criterion function:

$$D_R(p) = T \times \Sigma_{j=1}^{K-p} \psi_j \left(\hat{\Sigma}_{ff} \hat{\mathrm{B}}' Q_N (Q_N \hat{\Sigma}_{\varepsilon\varepsilon} Q_N)^+ Q_N \hat{\mathrm{B}} \right) - \ln(T) \times (N-1-p)(K-p) ,$$

where p = 1, ... , K–1, and $D_R(K)=0$.[5] Then, the RBIC estimator of the demeaned beta matrix ($Q_N\mathrm{B}$) equals the minimizer of $D_R(p)$. We refer to this estimator as the RBICD estimator. Note that even for the cases in which $\hat{\Sigma}_{\varepsilon\varepsilon}$ has full column rank, $Q_N\hat{\Sigma}_{\varepsilon\varepsilon}Q_N$ does not. That is why we use the Moore–Penrose generalized inverse of $Q_N\hat{\Sigma}_{\varepsilon\varepsilon}Q_N$ for $D_R(p)$.

Because the RBIC estimator is consistent as $T\to\infty$ with fixed N, we can expect the estimator to have good finite-sample properties for data with large T and relatively small N. However, to our knowledge, no asymptotic results are available from which the finite-sample properties of the RBIC estimator can be deduced for the cases in which both T and N are large. Thus, in the next section, we conduct Monte Carlo simulations to investigate the finite sample properties of the RBIC and RBICD estimators for the cases in which both N and T are large.

III. Finite-Sample Properties of RTC and RBIC Estimators

A. Simulation Setup

The foundation of our simulation exercises is the following data-generating process:

$$x_{it}=\alpha_i+\Sigma_{j=1}^{K}\beta_{ij}f_{jt}+\varepsilon_{it};\ \varepsilon_{it}=\phi\frac{(\xi_{i1}h_{1t}+\xi_{i2}h_{2t})}{\sqrt{\xi_{i1}^2+\xi_{i2}^2}}+\sqrt{1-\phi^2}v_{it},$$

where the f_{jt}, ξ_{i1}, ξ_{i2}, h_{1t}, h_{2t} and v_{it} are all independent random variables from N(0,1), and ϕ is a parameter between 0 and 1. For simplicity, we set $\alpha_i=0$ for all i. The f_{jt} represent empirical factors. The variance of the error ε_{it} is fixed at 1 for all i and t. The unobserved factor

[5] Let P_N be an $N\times(N-1)$ orthonormal matrix such that $P_N'1_N=0_{(N-1)\times 1}$ and $P_NP_N'=Q_N$. Then, the $D_R(p)$ statistic can be computed with P_N instead of Q_N. One advantage of this replacement is that $P_N'\hat{\Sigma}_{\varepsilon\varepsilon}P_N$ is invertible while $Q_N\hat{\Sigma}_{\varepsilon\varepsilon}Q_N$ is not.

components h_{1t} and h_{2t} in ε_{it} can be viewed as common factors that are not correlated with the empirical factors f_{jt}. The errors ε_{it} are cross-sectionally correlated by h_{1t} and h_{2t} if $\phi \neq 0$. For the reported simulations, we set ϕ = 0.2. Using greater values for ϕ only changes estimation results immaterially. We have also considered the cases in which the ε_{it} are serially correlated. We do not report the results because they are not materially different from the results reported below.

We generate the beta matrix B by the following 3 steps. First, we draw an $N \times r$ random matrix B^o such that its first column equals the vector of ones and the entries in the other columns are drawn from N(0,1). Second, we draw a random $K \times K$ positive definite matrix, compute the first r orthonormalized eigenvectors of the matrix and set a $K \times r$ matrix C using the eigenvectors.[6] Finally, we set $\mathrm{B} = \mathrm{B}^o \Lambda^{1/2} C'$, where $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_r)$. Under this setup, the response variables x_{it} are generated by r *orthogonal* factors $f_t^o = (f_{1t}^o, ..., f_{rt}^o)' = \Lambda^{1/2} C' f_t$ with $\mathrm{var}(f_t^o) = \Lambda$. Our simulation setup allows the empirical factors in f_t to be correlated because each factor is a linear combination of the orthogonal factors in f_t^o. In addition, under this setup, $\mathrm{rank}(\mathrm{B}) = r$ and $\mathrm{rank}(\mathrm{Q}_N \mathrm{B}) = r - 1$.

Under our data-generating setup, each of the factors in f_t can have non-zero explanatory power for x_{it} even if B does not have full column rank. The parameter λ_j equals the variance of the *j*th orthogonal factor, f_{jt}^o. Given that B^o is drawn from N(0,1), the λ_j equals the signal-to-noise ratio (SNR) of f_{jt}^o (i.e., the ratio of the return variations caused by the orthogonal factor

[6] The random matrix is of the form $M'M$, where the entries of the $K \times K$ matrix M are drawn from N(0,1).

f_{jt}^{o} and by the error ε_{it}). The average R^2 (average explanatory power of the factors in f_t for individual response variables x_{it}) equals $(\Sigma_{j=1}^{r}\lambda_j)/(1+\Sigma_{j=1}^{r}\lambda_j)$.

We try 4 values of *T*: *T* = 60, 120, 240 and 480. For each *T*, we generate up to 6 different sets of response variables *N* such that *N* < *T*.[7] We try *N* = 25, 50, 75, 100, 200 and 400. For each combination of *N* and *T*, we also consider 2 cases: one with 5 empirical factors (*K* = 5) and the other with 15 factors (*K* = 15). For both cases, we use 2 different values for rank(B), $r=1$ and 4, and 3 different values, $\lambda =$ 0.03, 0.1 and 0.5, for the SNR of an orthogonal factor. When $r=4$, we use the same value of λ for all 4 factors: $\lambda_j=\lambda$ for $j=1, 2, 3$ and 4. For each combination of *N*, *T*, *K*, *r* and λ, we generate 1,000 samples.[8]

We investigate the finite-sample performances of the RTC, RBIC and RBICD estimators. The RTC estimator is consistent under our setup. We do not separately consider the alternative

[7] We do so because $\hat{\Sigma}_{\varepsilon\varepsilon}$ is not invertible if $N+K+1>T$.

[8] Our simulation setup may not represent the true data-generating processes of asset returns. However, we choose parameter values such that the simulated data have properties similar to those of actual return data. First, empirical studies of asset pricing models routinely use monthly data over 5-, 10-, 20-, or 40-year data. The values of *T* are chosen to be consistent with this practice. Second, the empirical factors proposed in the literature sometimes have low explanatory power for stock returns. To investigate the cases in which empirical factors have limited explanatory power for response variables, we generate data in which the orthogonal factors f_t^o have very low SNRs, $\lambda=0.03$. Third, the idiosyncratic components of actual returns are likely to be cross-sectionally correlated. Under our simulation setup, the error terms are cross-sectionally correlated by the unobserved factor components h_{1t} and h_{2t}.

TC estimators suggested by Robin and Smith (2000) and Kleibergen and Paap (2006) because they are numerically the same as the RTC estimator if they are computed with $\hat{\Omega}_R$.

B. Simulation Results

We begin by considering the performance of the RTC estimator. Table 1 reports the RTC estimation results from the data simulated with 5 and 15 empirical factors (*K* = 5 and *K* = 15). We consider 2 cases: $r = 1$ for *K* = 5 and $r = 4$ for *K* = 15. Data are generated such that the orthogonal factors in f_t^o have SNRs of 0.03, 0.1 and 0.5. Table 1 reports the percentages (%) of correct estimation. The percentages of underestimation and overestimation are reported below in parentheses.

Table 1 shows that the RTC estimator performs rather poorly for most of the (T, N) combinations. For the cases with *N* = 25 and $T \leq 120$, the accuracy of the RTC estimator is not greater than 50%. The estimator's performance deteriorates as *T* decreases or *N* increases. For example, for the cases with *N* = 25 and *T* = 480, the accuracy of the RTC estimator is between 87% and 90.1%. However, the accuracy of the estimator is not better than 80% for the cases with $N = 25$ and $T \leq 240$. For the cases with $N \geq 50$, the accuracy of the RTC estimator is not better than 71% regardless of *T*. For any *T*, as *N* increases to about $T / 2$, the accuracy of the RTC estimator drops to near 0. This pattern is observed from the simulations with any combination of *K* and *r*, whether empirical factors are weak ($\lambda = 0.03$) or strong ($\lambda = 0.5$). Furthermore, as shown in Tables 2 and 3, the RBIC estimator outperforms the RBIC estimator in all of the cases considered in Table 1.

Table 2 reports the performance of the RBIC estimator for the cases with 5 empirical factors (*K* = 5). The data-generating process is the same as the one described in the beginning of this section. The accuracy of the RBIC estimator appears to have a non-monotonic relationship with the number of response variables (*N*). For the cases with $r = 1$, the accuracy of the estimator increases with N when $N \leq T/2$ and decreases with N when $T/2 < N < T$. The estimator overestimates the rank of the beta matrix when $T/2 < N < T$. We find a similar pattern for the cases with $r = 4$. The accuracy of the RBIC estimator increases with N up to the points where $T/2 < N \leq 2T/3$. However, the accuracy drops quickly with N passing those points. These results suggest that the RBIC estimator should be used with caution for data with $N > T/2$.[9]

For the cases with $T \geq 240$ and $N \leq T/2$, the accuracy of the RBIC estimator is near 100% except for the cases with $\lambda = 0.03$. Not surprisingly, the RBIC estimator tends to underestimate the rank of the beta matrix when empirical factors have very low SNRs. However, the accuracy of the RBIC estimator improves with N as long as $N \leq T/2$. For example, as shown in panel (b) of Table 2, for the cases with $T = 240$ and $\lambda = 0.03$, the RBIC estimator's accuracy improves from 33.3% to 100% as N increases from 25 to 100. We observe a similar pattern in the cases with $T = 480$ and $\lambda = 0.03$.

The accuracy of the RBIC estimator may depend on the number of empirical factors (K). Table 3 reports the results from the cases with K = 15. Comparing the results in Tables 2 and 3, we can see that the accuracy of the RBIC estimator generally falls as K increases without changing factors' overall explanatory power (R^2). The performance pattern of the RBIC estimator reported in Table 3 is almost identical to that in Table 2. Even when the explanatory power of empirical factors is very low ($\lambda_j = 0.03$), the RBIC estimator is quite accurate if T or N

[9] From some unreported simulations, we found that when $T \geq 960$, the RBIC estimator is accurate as long as $N \leq 0.8 \times T$. However, for data with $T \leq 480$, the condition $N \leq T/2$ seems to be important to secure the reliability of the RBIC estimator.

are large, as long as $N \le T/2$. For example, when $\lambda = 0.03$ and $r = 4$, the accuracy of the RBIC estimator is 99.9% when $(T, N) = (240,100)$ and 98.4% when $(T, N) = (480,50)$.

Finally, we consider the performance of the RBICD estimator. Under our simulation setup, the rank of the demeaned beta matrix equals $r - 1$. We use the simulated data with $r = 4$. The results, reported in Table 4, are similar to those reported in Tables 2 and 3. The accuracy of the RBICD estimator increases with N when $N \le T/2$ and also increases monotonically with T.

There are 3 main results from our simulation exercises. First, the RTC estimator is generally inaccurate unless T is very large (e.g., $T = 480$) and N is sufficiently small compared to T (e.g., $T = 480$ and $N \le 25$). Second, the RBIC estimator outperforms the RTC estimator in most of the cases we consider. Third, the accuracy of the RBIC estimator has a non-monotonic relationship with the number of response variables (N). The power of the estimator initially increases with N but falls as N increases. For the data with N close to T, we cannot expect reliable inferences. The RBICD estimator shows the same pattern as the RBIC estimator.

IV. Application

In this section, we apply the RBIC and RBICD methods to U.S. stock return data and various empirical factors proposed in the literature. We use monthly and quarterly stock portfolio returns and monthly individual stock returns from Jan. 1972 to Dec. 2011.

For the analysis using monthly data, we consider 17 non-repetitive empirical factors from 6 different models and 2 reversal factors. Specifically, we use the 3 factors of Fama and French (1993) (FF3), the 4 factors of Carhart (1997) (Carhart), the 5 factors of Fama and French (2015) (FF5), the 4 factors of Hou et al. (2015) (HXZ), the 5 factors of Chen et al. (1986) (CRR), and the 3 factors of Jagannathan and Wang (1996) (JW).

The FF3 factors are the excess market return (VW: the return on the CRSP value-weighted portfolio minus the return on the 1-month Treasury bill), Small Minus Big (SMB) and High Minus Low (HML) factors.[10] The Carhart factors are the FF3 plus the momentum factor (MOM, selling losers and buying winners 6 to 12 months ago). The FF5 factors are the FF3, Robust Minus Weak (RMW) and Conservative Minus Aggressive (CMA) factors. The HXZ factors are the VW, SMB, Investment-to-Asset (IA) and Return on Equity (ROE) factors. The CRR factors are 5 macroeconomic variables: industrial production (MP), unexpected inflation (UI), change in expected inflation (DEI), the term premium (UTS) and the default premium (UPR).[11] The JW factors are the VW, LAB (growth rate of labor income) and PREM (1-month lagged yield spread between BAA- and AAA-rated bonds) factors.[12] Finally, we include the 2 reversal factors (REV) to address the momentum effects discussed in Jegadeesh and Titman (1993): the REV_S (selling winners and buying losers 1 month ago) and REV_L (selling winners and buying losers 13 to 60 months ago) factors.

A. Estimation with Portfolio Return Data

[10]All of the FF factors are available from Kenneth French's Web site (www.dartmouth.edu/~kfrench/).

[11] The CRR factors are available from Laura Xiaolei Liu's Web page (http://www.bm.ust.hk/~fnliu/research.html). For detailed information on how these factors have been constructed, see Liu and Zhang (2008).

[12]The LAB factor is constructed using the NIPA 2.1 and NIPA 2.6 tables for quarterly and monthly data, respectively. The tables are available at the Bureau of Economic Analysis Web page (http://www.bea.gov/iTable). Specifically, the factor is the growth rate of total personal income minus personal dividend income divided by total population. The PREM factor is the 1-month lagged Moody's Seasoned Baa Corporate Bond Yield relative to Yield on the 10-Year Treasury Constant Maturity (BAA10Y) retrieved from FRED, Federal Reserve Bank of St. Louis (https://fred.stlouisfed.org/series/BAA10Y).

We use 6 sets of portfolio returns as response variables. Five of them are the 25 Size and Book-to-Market (BM) portfolios; 30 Industrial portfolios; 25 Size and Momentum portfolios; 25 Operating Profitability and Investment portfolios and 32 Size, Operating Profitability and Investment portfolios. Following the suggestion of Lewellen et al. (2010), we also consider the combined set of the 25 Size and BM and 30 Industrial portfolios.[13] The excess return on each portfolio is computed using the 1-month Treasury bill rate as the risk-free rate.

For sensitivity analysis, we also estimate the above factor models using quarterly observations. Analyzing quarterly portfolio returns, we can examine 5 additional factor models: the consumption CAPM (CCAPM), the conditional CCAPM of Lettau and Ludvigson (2001) (LL), the durable-consumption CAPM of Yogo (2006) (Yogo), the conditional CAPM of Santos and Veronesi (2006) (SV), and the investment-based CAPM of Li et al. (2006) (LVX). The empirical factors used by these models are CG (aggregate consumption growth rate) for the CCAPM; CG, CAY (1-period lagged aggregate consumption-to-wealth ratio) and CG×CAY for the LL model; VW, DCG (durable-consumption growth rate) and NDCG (nondurable-consumption growth rate) for the Yogo model; VW and VW×LC (labor income-to-consumption ratio) for the SV model; and DHH (change in the gross private investment for households), DCORP (change in gross private investment for non-financial corporate firms) and DNCORP (change in gross private investment for non-financial non-corporate firms) for the LVX model.[14]

[13] In unreported experiments, we also examined many other larger sets of portfolio returns by merging 2 of the 6 portfolio sets. We do not report the results from these experiments because they are very similar to those from this merged set of portfolio returns.

[14] We are grateful to Jonathan Lewellen and Stefan Nagel for sharing their data with us. The CG, CAY and LC factors can be directly downloaded or constructed using the data available from Sydney Ludvigson's website

Note that most of the factors used by these models, as well as some in the CRR and JW models, are non-traded factors. As is well known, and as we observe in our empirical results, non-traded factors have significantly lower explanatory power for returns than traded factors do.

1. Results from Monthly Stock Portfolio Returns

In this subsection, we report the estimation results obtained using the 6 sets of monthly portfolio returns as response variables. The estimation results from the data in the entire sample period are reported in Panel A of Table 5. Panels B and C report the estimation results using data from the first and second halves of the sample period, respectively. For each combination of portfolio sets and empirical factors, we report the adjusted R^2 ($\bar{R}^2$ to measure the explanatory power of empirical factors) and the estimated rank of the beta matrix by the RBIC estimator. The RBICD estimation results are reported in parentheses. Our simulation results, reported in Section III.B, indicate that unless some factors have too low SNRs, the RBIC estimator produces reliable inferences when data with $T \geq 240$ and $N \leq T/2$ are used. The sizes of the data used for Table 5 satisfy these conditions.

The main observations from the RBIC estimation results in Panel A of Table 5 are the following. First, for any of the 6 portfolio sets, the RBIC estimator predicts that the beta matrix

---

(http://www.econ.nyu.edu/user/ludvigsons). The DCG and NDCG factors are constructed using data from the *NIPA* 2.3.3 and *NIPA* 2.3.5 tables. We also use the *Consumer-Durable Goods: Chain-Type Quantity Indexes for Net Stock* table for constructing DCG. All of these tables are available at the Bureau of Economic Analysis Web page (http://www.bea.gov/iTable). For the DHH, DCORP and DNCORP factors we use the *Flow of Funds Accounts* tables available at the Federal Reserve Board's Web page (http://www.federalreserve.gov). Specifically, we use table FA155019005 for the DHH factor, tables FA105019005 and FA105020005 for the DCORP factor and tables FA115019005 and FA115020005 for the DNCORP factor.

corresponding to the FF3 factors has full column rank ($r = 3$). We obtain a similar result for the HXZ model. For 5 of the 6 portfolio sets, the HXZ model produces full column beta matrices ($r = 4$). One exception is the 25 Operating Profitability and Investment portfolios, for which the beta matrix has a rank of 3 ($r = 3$).

Second, 2 other models occasionally generate full column rank beta matrices. The Carhart model generates full column rank beta matrices for 2 portfolio sets. The FF5 model produces a full column rank beta matrix only for the 32 Size, Operating Profitability and Investment portfolios. For other sets of portfolios, the beta matrices corresponding to the FF5 factors are all rank-deficient: The estimated ranks are usually 4.

Third, the beta matrices from the MOM plus REV, CRR and JW models are estimated to be all rank-deficient, regardless of which portfolio set is analyzed.[15] The explanatory power of these factors for portfolio returns is not as strong as that of the factors in the FF3, Carhart, HXZ or FF5 models. The explanatory power of the CRR factors is particularly low. The factors explain no more than 2% of the average total variation in the portfolio returns analyzed.

Fourth and finally, when all of the 17 empirical factors are used in regression, the estimated rank of the beta matrix is 4 or 5, depending on the portfolio sets analyzed. In other words, risk premiums can be identified for only up to 4 or 5 factors among the 17 factors. This means that other than the FF3 factors, 1 or 2 additional factors have identifiable risk prices (zeros or non-zeros) and drive co-movement in stock returns. Possible candidates for these additional factors are the MOM factor (in the Carhart model), the RMW and CMA factors (in the FF5 model), or the IA and ROE factors (in the HXZ model). Table 5 shows that when these 5

---

[15] Similar to our analysis, Kleibergen and Paap (2006) also find that the beta matrix from the JW model does not have full column rank.

empirical factors are added to the FF3 model (an 8-factor model), the beta matrix has exactly the same rank as the beta matrix from all of the 17 empirical factors. This is true for all of the 6 portfolio sets.

In Panel A of Table 5, nearly 80% of the RBICD estimates are equal to their counterpart RBIC estimates. For instance, for any factor model estimated with the data from the 32 Size, Operating Profitability, and Investment Portfolios, the rank of the demeaned beta matrix has the same rank as the beta matrix. In contrast, for the 25 Size and Momentum Portfolios, the RBICD estimates are often smaller than the RBIC estimates by 1, suggesting that one beta (or one linear combination of different betas) may be constant over different portfolio returns. However, for other sets of portfolios, the RBIC and RBICD estimation results are generally similar.

Panels B and C of Table 5 report the rank estimates obtained using the data from 2 subsample periods: 1972–1991 and 1992–2011. Each of these subsamples has $T = 240$. Comparing the results reported in all 3 panels, we can observe 2 things. First, the estimated ranks of beta matrices from the second subsample are similar to those from the entire data set. Second, the rank estimates from the second subsample are generally greater than those from the first subsample. In contrast, the reported adjusted R^2 show that the empirical factors we consider appear to have stronger explanatory power for portfolio returns in the first subsample period than in the second subsample period. In order to obtain better insights about these patterns of rank estimates and adjusted R^2 over time, we have conducted some unreported analyses. We found that the explanatory power of the excess market return (VW) gets weaker over time. Consequently, the explanatory power of the models including the excess market return as a factor gets weaker too. In contrast, the explanatory power of some other traded factors becomes stronger, although not enough to offset the decrease in the explanatory power of the excess market return. As shown in Section III.B, the RBIC method tends to underestimate the rank of a beta matrix when the data with $T = 240$ are used and some factors have very low SNRs. This may explain why the estimated ranks of beta matrices are smaller in the first subsample period.

2. Results from Quarterly Stock Portfolio Returns

Using quarterly portfolio data, we re-estimate the FF3, Carhart, FF5 and HXZ models, which most often produce full column rank beta matrices for monthly portfolio return data. We consider the same 6 sets of portfolios as in the previous subsection. The sample period is from the first quarter of 1972 to the fourth quarter of 2011 ($T = 160$). The estimation results are presented in Table 6.[16]

We find again that the beta matrix of the FF3 factors has full column rank for all of the 6 portfolio sets. In fact, the FF3 model is the only model that has the full column rank beta matrix for all 6 sets of portfolios. The HXZ and Carhart models produce full column rank beta matrices only for 1 set of portfolios. The FF5 factors does not produce a full rank beta matrix in this data set. In contrast, for the 8-factor model with the FF5, MOM, IA and ROE factors, the estimated beta matrix rank is 4 for 5 out of the 6 portfolio sets.

Next, we consider 5 additional models with macroeconomic factors: the unconditional CCAPM and the models of Lettau and Ludvigson (2001) (LL), Yogo (2006) (Yogo), Santos and Veronesi (2006) (SV), and Li et al. (2006) (LVX). Because all of the factors used by these models are available only quarterly, we refer to them as *quarterly macroeconomic factors*. Similar to the CRR and JW factors (other than the VW factor), these factors are non-traded factors.

[16] We do not report the results from data in subsample periods because our simulation results indicate the data with $T < 160$ are unlikely to produce reliable results since we have only 160 quarterly observations for each asset.

The results obtained by estimating these models are reported in Table 6. The beta matrices from the CCAPM, LL and LVX models have rank of zero for all 6 portfolio sets. The RBIC and RBICD estimates from Yogo's 3-factor model are 1 for all 6 portfolio sets. These results are not surprising given that the quarterly macroeconomic factors have very low explanatory power for stock returns. None of the CCAPM, LL, LVX and Yogo models appear to be able to identify risk premiums. The SV model generates a full column rank beta matrix ($r = 2$) for only 1 of the 6 portfolio sets.

When we add all of the quarterly macroeconomic factors to the 8-factor model of the FF5, MOM, IA and ROE factors, the estimated rank of the beta matrix does not increase for any of the 6 portfolio sets. It appears that the macroeconomic factors do not contain additional information.

B. Estimation with Individual Stock Return Data

In this subsection, we report the rank estimation results obtained using monthly returns of individual stocks as response variables. Individual stock returns include dividends. The data are downloaded from CRSP and exclude REITs (Real Estate Investment Trusts) and ADRs (American Depositary Receipts). We also exclude the observations in which more than 300% excess returns occur in a given month because such huge variations are unlikely due to changes in common factors. Excessively high or low returns are most likely to be driven by idiosyncratic shocks.

Table 7 shows the estimation results from individual stock returns. We sample 10,000 random sets of 50 stocks (N = 50) for each data set in Panel A and 10,000 random sets of 200 stocks (N = 200) in Panel B. This exercise gives us further information regarding how the RBIC and RBICD estimators would perform when the SNRs of empirical factors are low or when N is greater than $T/2$. Table 7 reports the averages of the RBIC and RBICD estimates and adjusted

R^2 from 10,000 sets of randomly chosen individual stocks with their standard deviations (in parentheses).

Panel A of Table 7 reports the estimation results from the data with $N < T/2$. Comparing the reported adjusted R^2 in Table 5 and Panel A of Table 7, we can see that empirical factors have much stronger explanatory power for portfolio returns than for individual stock returns. For each factor model, the adjusted R^2 from portfolio returns are at least 3 times larger than those from individual returns. Thus, it is expected that rank estimates would be smaller when individual returns are used in regressions. The results reported in Panel A of Table 7 confirm this expectation. The rank estimates are, on average, smaller than 3 even if we use all of the available seventeen monthly empirical factors (non-redundant factors in the FF5, HXZ, CRR and JW models and MOM and REV factors). Except for the FF3 model, the average ranks of beta matrices from other models are substantially smaller than the numbers of factors used in the models.

Panel B of Table 7 shows the effect of using data with $N > T/2$ on rank estimation. Notice that $N < T/2$ for the entire sample period (1972 – 2011), while $N > T/2$ for the 2 subsample periods. The results from the entire data set are similar to those in Panel A of Table 7. However, for both of the subsamples, the rank estimate is close to the total number of empirical factors used in regression. These results are consistent with the results from our simulations with data sizes of $N \leq T/2$ and $N > T/2$. Even for actual data analysis, it seems prudent to use the data with $N \leq T/2$ for estimating the rank of the beta matrix.

C. Selecting Factors

When a factor model does not produce a beta matrix with full column rank (that is, $r < K$), researchers may want to identify the r factors that can produce a beta matrix whose rank also equals r (full column rank) so that the risk prices of the r factors are identifiable. Burnside

(2016) proposed a method to construct *r* statistical factors, which are linear combinations of the original *K* empirical factors. However, a limitation of Burnside's approach might be that the resulting statistical factors are difficult to interpret.[17]

A sequential elimination method can be used to find a subset of empirical factors with a full column rank beta matrix. To be precise, suppose that the initial RBIC estimate ($\hat{r}_0$) from a set of *K* empirical factors is smaller than *K* ($\hat{r}_0 < K$). The first step of the sequential method is to re-estimate the rank of the beta matrix after removing 1 factor from the set of *K* empirical factors. If the estimated rank of the beta matrix with the remaining $(K-1)$ factors remains unchanged at $\hat{r}_0$, it can be concluded that the removed factor lacks power to identify its risk price. The removed factor needs not be reinstated in rank estimation. Conversely, if the rank estimate drops, the removed factor can be viewed as the one whose price can be identified (zero or non-zero). Then, the removed factor should be reinstated. After determining whether the initially removed factor should be reinstated or not, the second step of the sequential method is to repeat the first step with another factor removed. Repeat these steps until the estimated rank of the beta drops whenever any of remaining factors is removed from estimation.

One pitfall of this sequential procedure is that its outcome may depend on the order by which each empirical factor is tested for elimination. This order dependence problem is likely to arise when multiple empirical factors are correlated with the same true common factors. As an example, suppose that returns are generated by 2 true common factors (say, g_1 and g_2) and that

[17] Gospodinov, Kan and Robotti (2014) proposed a method that selects factors whose prices are non-zeros. They called these factors "relevant" to distinguish them from "irrelevant" factors: useless factors and factors whose prices are zero.

3 empirical factors (say, f_1, f_2 and f_3) are used in regression. Suppose that $f_1 = g_1$, while f_2 and f_3 are both correlated with g_2. For this case, the beta matrices of the 3 empirical factors and the 2 pairs, (f_1, f_2) and (f_1, f_3), all have a rank of 2. Thus, the sequential method may result in 2 different sets of 2 factors depending on the order of testing each variable: (f_1, f_2) if f_3 is tested ahead of f_2; and (f_1, f_3) if f_2 is tested ahead of f_3.

When multiple sets of $\hat{r}_0$ factors result as in the above example, it may not be clear how the identified factors should be interpreted. However, as noted in Lewellen et al. (2010), any of the multiple sets of $\hat{r}_0$ factors can explain individual expected returns as long as the empirical factors in each set are correlated with all of the true factors that price individual risky assets. For this case, the sequential procedure is useful because any set of $\hat{r}_0$ empirical factors with full rank beta matrices may explain individual expected returns equally well.

If a researcher wants to find all of the possible sets of $\hat{r}_0$ factors leading to full column rank beta matrices, she/he can estimate the rank of the beta matrix for each of all possible subsets of $\hat{r}_0$ factors.[18] However, the rank estimation alone cannot single out the best model among the $\hat{r}_0$-factor models found by this method. Further asset pricing tests beyond the rank estimation are needed. For example, the preferred model may be determined by comparing the performances of different models in terms of a specific metric such as a factor model's average absolute pricing error.

[18] The number of all possible combinations of $\hat{r}_0$ can be very large if the number of empirical factors is large. For example, in our case with the 26 empirical factors, there are 65,780 sets of 5 factors.

The sequential method might not always successfully select $\hat{r}_0$ factors with full column rank beta matrices. It is possible that the method fails to select such $\hat{r}_0$ empirical factors even if many different test orders are tried. This failure is likely to happen when some of the *K* empirical factors are only weakly correlated with true factors. For example, consider the above 3-factor example again. Suppose that each of f_2 and f_3 is only weakly correlated with g_2: That is, both f_2 and f_3 have very low SNRs. For this case, the RBIC method may underestimate the rank of the beta matrix corresponding to either (f_1, f_2) or (f_1, f_3). However, if f_2 and f_3 jointly have sufficiently high explanatory power for g_2, use of all 3 empirical factors may result in a beta matrix whose rank can be estimated to be 2.

To demonstrate how the sequential method works with actual data, we consider 2 factor models: the FF5 model and the 8 factor model that uses the MOM factor and the non-redundant factors in the FF5 and HXZ models. The response variables are the monthly returns of the 30 Industrial Portfolio plus the 25 Size and Book to Market portfolios. We use the data available for the entire sample period from Jan. 1972 to Dec 2011. The 2 models are predicted to have beta matrices with ranks equal to 4 and five, respectively (see Panel A of Table 5).

The sequential method applied to the FF5 model shows that the estimated beta matrix rank remains at 4 even if the CMA factor is removed. In contrast, removing any other factor reduces the rank of the beta matrix to 3. Consequently, the CMA factor appears to be a useless factor, while the VW, SMB, HML and RMW factors are possibly priced factors. For the 8 factor model, the sequential method selects 5 factors that generate full column beta matrix: the VW, SMB, HML, RMW and ROE factors.

In unreported experiments we also applied the sequential method to other sets of portfolio returns and other factor models. The method did not always produce clear-cut results. For example, for some sets of portfolios and some factor models that have rank deficient beta matrices ($\hat{r}_0 < K$), the sequential method was unable to identify a subset of $\hat{r}_0$ factors that generate a full column rank beta matrix.

D. Summary of Empirical Results

The main results from our analysis of actual return data can be summarized as follows. First, for the portfolio returns and the different sample periods we consider, the FF3 factors always generate full column rank beta matrices. The HXZ model of 4 factors produces full column rank beta matrices quite often (but not always). Many other models rarely produce full column rank beta matrices. Second, when all of the 26 empirical factors (including the quarterly macroeconomic factors) are used in regression, the estimated ranks of beta matrices are at maximum 6 when the U.S. portfolio returns are used as response variables. Third, the rank estimates are on average smaller when individual stock returns are used as response variables because empirical factors have much weaker explanatory power for individual stock returns than for portfolio returns. Forth, the estimated ranks of beta matrices in the earlier period of 1972 – 1991 are equal to or smaller than those in the later period of 1992 – 2011. Fifth and finally, the sequential method introduced in section IV.C can be used to select the factors whose risk prices are identifiable (zero or nonzero). A factor model may contain 2 types of empirical factors: the factors whose prices are identifiable (to be zeros or non-zeros) and those whose prices are not identifiable (e.g., useless factors). The sequential method can be used to select the former group of factors although it is not always successful.

V. Concluding Remarks

In this paper we have considered estimation methods for the rank of a beta matrix corresponding to a multifactor model. We have conducted Monte Carlo simulations to determine which of the existing rank estimators produces the most-reliable inferences, particularly when the number of assets (*N*) to be analyzed is large. Our simulation results indicate that Testing Criterion estimators may be reliable only if the number of time series observations (*T*) is very large and the number of assets (*N*) is substantially small. In contrast, the restricted Bayesian Information Criterion (RBIC) estimator proposed by Cragg and Donald (1997) is found to be quite accurate for the data with $T \geq 240$ and $N < T / 2$.

In our empirical analysis with the RBIC method, the 3-factor model of Fama and French (1993) is the only model that consistently produces full column rank beta matrices over different data. The 4-factor model of Hou et al. (2015) also generates full rank beta matrices quite often, especially for the portfolio return data from more recent time periods. The other factor models considered often fail to produce full column rank beta matrices.

Brown (1989) found that the behavior of the eigenvalues of the covariance matrix of the U.S. stock returns is consistent with either 1 common factor or *K* equally important factors. More recently, Bai and Ng (2002) and Onatski (2010) developed estimation methods that can estimate the number of common factors without using empirical factors. Using these methods, they found evidence that 2 common factors drive U.S. individual stock returns. Our analysis using individual stock returns yields similar results. In contrast, our results from portfolio return data suggest that five factors drive the co-movement in stock returns and that the explanatory power of each of the factors is not equal. It is not surprising that the results from portfolio and individual stock returns suggest different numbers of common factors. Individual returns are exposed to large idiosyncratic risks, and thus, factors have low signal-to-noise ratios. Asset pricing models can be better tested with portfolio returns because they are more highly correlated with common factors.

Appendix. The Consistency of the RBIC Estimator

In this appendix, we show that the RBIC estimator is consistent. A similar proof can be used to show the consistency of the RBICD estimator. We begin with the following assumptions. All of the asymptotic assumptions and results here are for the cases in which $T \to \infty$ with fixed *N*.

Assumption A (factors): $T^{-1}\Sigma_{t=1}^{T}(f_t - \bar{f})(f_t - \bar{f})' \to_p \Sigma_{ff}$ and $\bar{f} \to_p \mu_f$, where Σ_{ff} is a finite and positive definite matrix and μ_f is a finite vector.

Assumption B (idiosyncratic errors): For given fixed *N*, $E(\varepsilon_t) = 0_{N\times 1}$ and $T^{-1}\Sigma_{t=1}^{T}\varepsilon_t\varepsilon_t' \to_p \Sigma_{\varepsilon\varepsilon}$, where $\Sigma_{\varepsilon\varepsilon}$ is an $N\times N$ finite and positive definite matrix.

Assumption C (factors and idiosyncratic errors): $T^{-1/2}\Sigma_{t=1}^{T}\varepsilon_t \otimes \left(1 \quad f_t'\right)' \to_d N(0_{N(K+1)\times 1}, \Xi)$, where Ξ is a finite and positive definite matrix.

Assumption D (betas): For fixed $N > K$, $\text{rank}(\mathrm{B}) = r$, where $0 \le r \le K$.

Assumptions A–C are the standard assumptions under which the OLS estimator of B is consistent and asymptotically normal. Chapters 3 and 4 of White (1984) provide detailed conditions under which the 3 assumptions hold. We note that the assumptions allow both factors and idiosyncratic errors to be heteroskedastic and/or autocorrelated over time. In addition, under the assumptions it is straightforward to show that in (2), $\Omega = (I_N \otimes \Sigma_{ff}^{-1})\Xi(I_N \otimes \Sigma_{ff}^{-1})$ and that $\hat{\Sigma}_{\varepsilon\varepsilon}$ is a consistent estimator of $\Sigma_{\varepsilon\varepsilon}$ for fixed *N*: $\hat{\Sigma}_{\varepsilon\varepsilon} \to_p \Sigma_{\varepsilon\varepsilon}$.

Lemma A1. Under Assumptions A–D, for some positive number a_p,

(A-1) $$\Pi_T(\hat{A}_p, \hat{M}_p, \hat{\Omega}_R)/T \to_p a_p, \text{ for } p < r;$$

(A-2) $$\Pi_T(\hat{A}_r, \hat{M}_r, \hat{\Omega}_R) = T \times \Sigma_{j=1}^{K-r}\hat{\psi}_j = O_p(1).$$

Proof of Lemma A1. When $p < r$, for any A_p and M_p matrices of full column rank, $\text{rank}(A_p M_p') \leq p < r$. Thus, for any A_p and M_p, $\mathrm{B} \neq A_p M_p'$. Consequently, $\text{vec}(\hat{\mathrm{B}}' - \hat{A}_p \hat{M}_p')$ $\to_p \xi_p$, where ξ_p is a non-zero constant vector. Then, (A-1) holds because, by Cragg and Donald (1997),

$$\Pi_T(\hat{A}_p, \hat{M}_p, \hat{\Omega}_R)/T = \text{vec}(\hat{\mathrm{B}}' - \hat{A}_p M_p')'(\hat{\Sigma}_{\varepsilon\varepsilon}^{-1} \otimes \hat{\Sigma}_{ff})\text{vec}(\hat{\mathrm{B}}' - \hat{A}_p M_p')' \to_p a_p,$$

where $a_p = \xi_p'(\Sigma_{\varepsilon\varepsilon}^{-1} \otimes \Sigma_{ff})\xi_p$. Finally, by Smith and Robin (2000), $T\Sigma_{j=1}^{K-r}\psi_j(\hat{\Sigma}_{ff}\hat{\mathrm{B}}'\Sigma_{\varepsilon\varepsilon}^{-1}\hat{\mathrm{B}})$ is asymptotically a weighted sum of $(N-r)(K-r)$ independent chi-squared random variables even if $\hat{\Sigma}_{ff}$ and $\hat{\Sigma}_{\varepsilon\varepsilon}$ are replaced by any conformable finite positive definite matrices. Thus, (A-2) holds.

Theorem A1. Let $\hat{r}$ be the RBIC estimator. Then, under Assumptions A – D, $\text{pr}(\hat{r} = r) \to 1$. That is, $\hat{r}$ is a consistent estimator of r.

Proof of Theorem A1. $\text{pr}(\hat{r} \neq r) = \text{pr}[C_R(r) - C_R(\hat{r}) > 0]$. This is so because $C_R(r) - C_R(\hat{r}) > 0$ if and only if $\hat{r} \neq r$. Thus, we can complete the proof by showing that $\text{pr}[C_R(r) - C_R(\hat{r}) > 0] \to 0$.

Let $m(p) = (N-p)(K-p)$. Then, for $\hat{r} > (<) r$,

$$m(\hat{r}) - m(r) = (N-\hat{r})(K-\hat{r}) - (N-r)(K-r) < (>) 0. \tag{A-3}$$

Suppose that $\hat{r} > r$. Then,

$$\begin{aligned} \text{pr}\left[C_R(r) - C_R(\hat{r}) > 0\right] &= \text{pr}\left[T\Sigma_{j=1}^{K-r}\hat{\psi}_j - T\Sigma_{j=1}^{K-\hat{r}}\hat{\psi}_j + \ln(T)(m(\hat{r}) - m(r)) > 0\right] \\ &\leq \text{pr}\left[T\Sigma_{j=1}^{K-r}\hat{\psi}_j + \ln(T)(m(\hat{r}) - m(r)) > 0\right] \to 0 \end{aligned} \tag{A-4}$$

because $T \times \Sigma_{j=1}^{K-r}\hat{\psi}_j = O_p(1)$ by Lemma 1 and $\ln(T)(m(\hat{r}) - m(r)) \to -\infty$ by (A-3). Therefore, $\text{pr}(\hat{r} > r) \to 0$. Similarly, if $\hat{r} < r$,

$$\text{pr}[C_R(r) - C_R(\hat{r}) > 0] = \text{pr}\left[\Sigma_{j=1}^{K-r}\hat{\psi}_j - \Sigma_{j=1}^{K-\hat{r}}\hat{\psi}_j + \frac{\ln(T)}{T}(m(\hat{r}) - m(r)) > 0\right] \to 0 \tag{A-5}$$

because $\ln(T)/T \to 0$, and $\Sigma_{j=1}^{K-r}\hat{\psi}_j - \Sigma_{j=1}^{K-\hat{r}}\hat{\psi}_j \to_p -\xi_{\hat{r}}'(\Sigma_{\varepsilon\varepsilon}^{-1} \otimes \Sigma_{ff})\xi_{\hat{r}} < 0$ by Lemma 1. Thus, by (A-4) and (A-5), $\text{pr}[C_R(r) - C_R(\hat{r}) > 0] \to 0$ whenever $\hat{r} \neq r$. This completes the proof.

Remark. We note that the above proof is almost identical to the proof of Theorem 3 in Cragg and Donald (1997). The only difference is that we use (A-2) for the proof. For their proof, Cragg and Donald use the fact that $\Pi_T(\hat{A}_r, \hat{M}_r, \hat{\Omega})$ is an asymptotically chi-squared random variable with the degrees of freedom equal to $m(p) = (N-p)(K-p)$. We add this proof to clarify that the proof requires $\Pi_T(\hat{A}_r, \hat{M}_r, \hat{\Omega})$ be only an asymptotically bounded random variable, not a random variable of a particular distribution.

References

Ahn, S. C.; M. F. Perez; and C. Gadarowski. “Two-Pass Estimation of Risk Premiums with Multicollinear and Near-Invariant Betas.” *Journal of Empirical Finance,* 20 (2013), 1–17.

Al-Sadoon, M. M. “A General Theory of Rank Testing.” Barcelona GSE Working Paper 750 (2015).

Anderson, T. W. “Estimating Linear Restrictions on Regression Coefficients for Multivariate Normal Distributions.” *Annals of Mathematical Statistics,* 22 (1951), 327–351.

Bai, J., and S. Ng. “Determining the Number of Factors in Approximate Factor Models.” *Econometrica,* 70 (2002), 191–221.

Brown, S. J. “The Number of Factors in Security Returns.” *Journal of Finance,* 44 (1989), 1247–62.

Burnside, C. “Identification and Inference in Linear Stochastic Discount Factor Models with Excess Returns.” *Journal of Financial Econometrics,* 14 (2016), 295–330.

Carhart, M. M. “On Persistence in Mutual Fund Performance.” *The Journal of Finance,* 52 (1997), 57–82.

Chamberlain, G., and M. Rothschild. “Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Asset Markets.” *Econometrica,* 51 (1983), 1281–1304.

Chen, N.; R. Roll; and S. A. Ross. “Economic Forces and the Stock Market.” *Journal of Business*, 59 (1986), 383–403.

Cragg, J. G., and S. G. Donald. “Inferring the Rank of a Matrix.” *Journal of Econometrics,* 76 (1997), 223–250.

Fama, E. F., and K. R. French. “Common Risk Factors in the Returns on Stocks and Bonds.” *Journal of Financial Economics,* 33 (1993), 3–56.

Fama, E. F., and K. R. French. “A Five-Factor Asset Pricing Model.” *Journal of Financial Economics,* 116 (2015), 1–22.

Gospodinov, N.; R. Kan; and C. Robotti. “Misspecification-Robust Inference in Linear Asset-Pricing Models with Irrelevant Risk Factors.” *Review of Financial Studies,* 27 (2014), 2139–2170.

Harvey, C. R.; Y. Liu; and H. Zhu. “... and the Cross-Section of Expected Returns.” *Review of Financial Studies*, 29 (2015), 5–68.

Hou K.; C. Xue; and L. Zhang. “Digesting Anomalies: An Investment Approach.” *Review of Financial Studies*, 28 (2015), 650–705.

Jagannathan, R., and Z. Wang. “The Conditional CAPM and the Cross-Section of Expected Returns.” *Journal of Finance*, 51 (1996), 3–53.

Jegadeesh, N., and S. Titman. “Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency.” *Journal of Finance,* 48 (1993), 65–91.

Kan, R., and C. Zhang. “Two-Pass Tests of Asset Pricing Models with Useless Factors.” *Journal of Finance,* 54 (1999a), 203–235.

Kan, R., and C. Zhang. “GMM Tests of Stochastic Discount Factor Models with Useless Factor.” *Journal of Financial Economics,* 54 (1999b), 103–127.

Kleibergen, F., and R. Paap. “Generalized Reduced Rank Tests Using the Singular Value Decomposition.” *Journal of Econometrics,* 133 (2006), 97–126.

Lewellen, J.; S. Nagel; and J. Shanken. “A Skeptical Appraisal of Asset Pricing Tests.” *Journal of Financial Economics,* 96 (2010), 175–194.

Lettau, M., and S. Ludvigson. “Resurrecting the (C)CAPM: A Cross-Sectional Test When Risk Premia Are Time-Varying.” *Journal of Political Economy,* 109 (2001), 1238–1287.

Li, Q.; M. Vassalou; and Y. Xing. "Sector Investment Growth Rates and the Cross Section of Equity Returns." *Journal of Business*, 79 (2006), 1637–1665.

Lintner, J. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *Review of Economic and Statistics*, 47 (1965), 13–37.

Liu, L. X., and L. Zhang. "Momentum Profits, Factor Pricing, and Macroeconomic Risk." *Review of Financial Studies,* 21 (2008), 2417–2448.

Lustig, H. N., and S. V. Nieuwerburgh. "Housing Collateral, Consumption Insurance, and Risk Premia: An Empirical Perspective." *Journal of Finance*, 60 (2005), 1167–1219.

Merton, R. "An Intertemporal Capital Asset Pricing Model." *Econometrica,* 41 (1972), 867–887.

Mossin, J. "Equilibrium in A Capital Asset Market." *Econometrica*, 35 (1966), 768–783.

Onatski, A. "Determining the Number of Factors from Empirical Distribution of Eigenvalues." *Review of Economic and Statistics*, 92 (2010), 1004–1016.

Robin, J. M., and R. J. Smith. "Tests of Rank." *Econometric Theory,* 16 (2000), 151–175.

Ross, S. A. "The Arbitrage Theory of Capital Asset Pricing." *Journal of Economic Theory*, 13 (1976), 341–360.

Santos, T., and P. Veronesi. "Labor Income and Predictable Stock Returns." *Review of Financial Studies,* 19 (2006), 1–44.

Sharpe, W. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance*, 19 (1964), 425–442.

Treynor, J. "Toward A Theory of Market Value and Risky Assets." Working Paper, available at SSRN: http://ssrn.com/abstract=628187 (1962).

White, H. *Asymptotic Theory for Econometricians*. San Diego, CA: Academic Press (1984).

Yogo, M. "A Consumption-Based Explanation of Expected Stock Returns." *Journal of Finance*, 61 (2006), 539–580.

TABLE 1

RTC Estimation of Beta Matrix

Reported are the percentages (%) of correct estimations by the RTC estimator from 1,000 simulated data sets. The percentages (%) of under- and overestimation by each estimator are reported in parentheses (•,•). Data are drawn by $x_{it} = \Sigma_{j=1}^{K} \beta_{ij} f_{jt} + \varepsilon_{it}$; $\varepsilon_{it} = \phi(\xi_{1i}h_{1t} + \xi_{2i}h_{2t}) / (\xi_{1i}^2 + \xi_{2i}^2)^{1/2} + (1-\phi^2)^{1/2} v_{it}$, where K = 5 and 15, $\phi = 0.2$, and f_{jt} (j = 1, ... , K), $h_{j't}$, $\xi_{j'i}$ ($j' = 1, 2$) and v_{it} are all randomly drawn from $N(0,1)$. For the beta matrix B , we draw an $N \times r$ random matrix B^o such that its first column equals the vector of ones and the entries in the other columns are drawn from $N(0,1)$. We also draw a random $K \times K$ positive definite matrix, compute the first r orthonormalized eigenvectors of the matrix and set a $K \times r$ matrix C using the eigenvectors. $\mathrm{B} = \mathrm{B}^o \Lambda^{1/2} C'$, where $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_r)$. The rank of the beta matrix equals 1 or 4 (r = 1, 4) for cases when K equals 5 and 15, respectively. RTC refers to the Testing Criterion estimator of Cragg and Donald (1997), which is computed under the assumption that the ε_{it} are i.i.d. over time (but cross-sectionally correlated).

T	N	(a) $K = 5$, $r = 1$, $\lambda_1 = \lambda$			(b) $K = 15$, $r = 4$, $\lambda_i = \lambda$ for $i = 1,...,4$		
		$\lambda = 0.03$ $R^2 = 0.029$	$\lambda = 0.10$ $R^2 = 0.091$	$\lambda = 0.50$ $R^2 = 0.333$	$\lambda = 0.03$ $R^2 = 0.107$	$\lambda = 0.10$ $R^2 = 0.286$	$\lambda = 0.50$ $R^2 = 0.667$
60	25	4.1% (0.0, 95.9)	2.9% (0.0, 97.1)	2.3% (0.0, 97.7)	1.9% (0.0, 98.1)	0.1% (0.0, 99.9)	0.0% (0.0, 100)
	50	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
120	25	48.5% (0.0, 51.5)	46.4% (0.0, 53.6)	45.2% (0.0, 54.8)	45.9% (3.9, 50.2)	26.1% (0.0, 73.9)	20.7% (0.0, 79.3)
	50	0.1% (0.0, 99.9)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
	75	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
	100	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
240	25	79.2% (0.0, 20.8)	77.5% (0.0, 22.5)	76.6% (0.0, 23.4)	74.3% (2.0, 23.7)	68.1% (0.0, 31.9)	66.1% (0.0, 33.9)
	50	27.5% (0.0, 72.5)	27.0% (0.0, 73.0)	27.2% (0.0, 72.8)	4.6% (0.0, 95.4)	3.8% (0.0, 96.2)	3.6% (0.0, 96.4)
	75	1.2% (0.0, 98.8)	1.2% (0.0, 98.8)	1.2% (0.0, 98.8)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
	100	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
	200	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
480	25	90.1% (0.0, 9.9)	89.9% (0.0, 10.1)	90.0% (0.0, 10.0)	90.1% (0.1, 9.8)	87.8% (0.0, 12.2)	87.0% (0.0, 13.0)
	50	70.8% (0.0, 29.2)	70.5% (0.0, 29.5)	70.2% (0.0, 29.8)	51.6% (0.0, 48.4)	49.3% (0.0, 80.7)	48.1% (0.0, 51.9)
	75	37.5% (0.0, 62.5)	37.2% (0.0, 62.8)	37.3% (0.0, 62.7)	8.3% (0.0, 91.7)	7.1% (0.0, 92.9)	6.8% (0.0, 93.2)
	100	8.4% (0.0, 91.6)	8.5% (0.0, 91.5)	8.4% (0.0, 91.6)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
	200	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
	400	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)

TABLE 2

RBIC Rank Estimation of Beta Matrix from Five actors

Reported are the percentages (%) of correct estimations by the RBIC estimator from 1,000 simulated data sets with five factors (K = 5). The percentages (%) of under- and overestimation by each estimator are reported in parentheses (•,•). Data are generated by the same processes that are used for Table 1.

T	N	(a) $K = 5$, $r = 1$, $\lambda_1 = \lambda$			(b) $K = 5$, $r = 4$, $\lambda_i = \lambda$ for $i = 1,...,4$.		
		$\lambda = 0.03$ $R^2 = 0.029$	$\lambda = 0.10$ $R^2 = 0.091$	$\lambda = 0.50$ $R^2 = 0.333$	$\lambda = 0.03$ $R^2 = 0.107$	$\lambda = 0.10$ $R^2 = 0.286$	$\lambda = 0.50$ $R^2 = 0.667$
60	25	72.8% (22.3, 4.9)	89.6% (0.10, 10.3)	88.3% (0.0, 11.7)	2.1% (97.9, 0.0)	79.9% (19.8, 0.3)	99.3% (0.0, 0.7)
	50	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	37.3% (0.4, 62.3)	23.7% (0.0, 76.3)	20.6% (0.0, 79.4)
120	25	65.4% (34.6, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	2.4% (97.6, 0.0)	96.6% (3.4, 0.0)	100% (0.0, 0.0)
	50	97.6% (2.1, 0.3)	99.6% (0.0, 0.4)	99.5% (0.0, 0.5)	43.9% (56.1, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	75	73.6% (0.0, 23.4)	70.7% (0.0, 29.3)	70.6% (0.0, 29.4)	97.9% (1.7, 0.4)	99.1% (0.0, 0.9)	98.9% (0.0, 1.1)
	100	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	28.8% (0.0, 71.2)	25.3% (0.0, 74.7)	23.9% (0.0, 76.1)
240	25	97.7% (2.3, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	33.3% (66.7, 0.0)	99.9% (0.1, 0.0)	100% (0.0, 0.0)
	50	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	81.8% (18.2, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	75	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	98.5% (1.5, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	100	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	200	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	36.7% (0.0, 63.3)	34.9% (0.0, 65.1)	34.5% (0.0, 65.5)
480	25	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	90.0% (10.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	50	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	75	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	100	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	200	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	400	0.4% (0.0, 99.6)	0.4% (0.0, 99.6)	0.4% (0.0, 99.6)	53.6% (0.0, 43.7)	55.6% (0.0, 44.4)	55.1% (0.0, 44.9)

TABLE 3

RBIC Estimation of Beta Matrix from Fifteen Empirical Factors

Reported are the percentages (%) of correct estimations by the RBIC estimator from 1,000 simulated data sets with fifteen factors (K = 15). The percentages (%) of under- and overestimation by each estimator are reported in parentheses (•,•). Data are generated by the same processes that are used for Table 1.

T	N	(a) K = 15, r = 1, $\lambda_1 = \lambda$			(b) K = 15, r = 4, $\lambda_i = \lambda$ for i = 1,...,4		
		λ = 0.03 R^2 = 0.029	λ = 0.10 R^2 = 0.091	λ = 0.50 R^2 = 0.333	λ = 0.03 R^2 = 0.107	λ = 0.10 R^2 = 0.286	λ = 0.50 R^2 = 0.667
60	25	18.7% (0.0, 80.9)	9.3% (0.0, 90.7)	6.9% (0.0, 93.1)	24.9% (72.2, 2.9)	59.6% (8.7, 31.7)	40.5% (0.0, 59.5)
	50	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.6% (0.0, 99.4)	0.0% (0.0, 100)	0.0% (0.0, 100)
120	25	45.1% (54.9, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	0.2% (99.8, 0.0)	85.7% (14.3, 0.0)	100% (0.0, 0.0)
	50	83.6% (1.0, 15.4)	80.7% (0.0, 19.3)	79.2% (0.0, 20.8)	51.4% (48.2, 0.4)	94.0% (0.0, 6.0)	92.3% (0.0, 7.7)
	75	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	3.5% (0.0, 96.5)	1.7% (0.0, 98.3)	1.1% (0.0, 98.9)
	100	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
240	25	80.4% (19.6, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	4.9% (95.1, 0.0)	99.2% (0.8, 0.0)	100% (0.0, 0.0)
	50	99.9% (0.1, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	62.7% (37.3, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	75	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	97.4% (2.6, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	100	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	99.9% (0.1, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	200	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
480	25	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	64.9% (35.1, 0.0)	99.9% (0.1, 0.0)	100% (0.0, 0.0)
	50	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	98.4% (1.6, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	75	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	100	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	200	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	400	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)

TABLE 4

RBIC Rank Estimation of Demeaned Beta Matrix

Reported are the percentages (%) of correct estimations by the RBICD estimators from 1,000 simulated data sets. The percentages (%) of under- and overestimation by each estimator are reported in parentheses (•,•). Data are generated by the same processes that are used for Table 1.

T	N	(a) K = 5, r = 4, $\lambda_i = \lambda$ for i = 1,...,4			(b) K = 15, r = 4, $\lambda_i = \lambda$ for i = 1,...,4		
		λ = 0.03 R^2 = 0.107	λ = 0.10 R^2 = 0.286	λ = 0.50 R^2 = 0.667	λ = 0.03 R^2 = 0.107	λ = 0.10 R^2 = 0.286	λ = 0.50 R^2 = 0.667
60	25	9.3% (90.7, 0.0)	87.9% (11.2, 0.9)	98.7% (0.0, 1.3)	44.7% (47.3, 8.0)	53.1% (3.7, 43.2)	35.4% (0.0, 64.6)
	50	8.0% (0.0, 92.0)	3.7% (0.0, 96.3)	3.2% (0.0, 96.8)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
120	25	8.3% (91.7, 0.0)	98.2% (1.8, 0.0)	100% (0.0, 0.0)	1.2% (98.8, 0.0)	89.9% (10.1, 0.0)	100% (0.0, 0.0)
	50	60.2% (39.8, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	67.4% (30.8, 1.8)	92.0% (0.0, 8.0)	90.8% (0.0, 9.2)
	75	96.2% (0.5, 3.3)	95.7% (0.0, 4.3)	94.6% (0.0, 5.4)	1.5% (0.0, 98.5)	0.8% (0.0, 99.2)	0.5% (0.0, 99.5)
	100	3.3% (0.0, 96.7)	2.6% (0.0, 97.4)	2.3% (0.0, 97.7)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
240	25	46.1% (53.9, 0.0)	99.9% (0.1, 0.0)	100% (0.0, 0.0)	11.7% (88.3, 0.0)	99.5% (0.5, 0.0)	100% (0.0, 0.0)
	50	88.1% (11.9, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	73.9% (26.1, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	75	98.9% (1.1, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	98.2% (1.8, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	100	99.9% (0.1, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	200	8.5% (0.0, 91.5)	7.6% (0.0, 92.4)	7.2% (0.0, 92.8)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)
480	25	92.9% (7.1, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	73.3% (26.7, 0.0)	99.9% (0.1, 0.0)	100% (0.0, 0.0)
	50	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	98.9% (1.1, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	75	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	100	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	200	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)	100% (0.0, 0.0)
	400	23.7% (0.0, 76.3)	23.4% (0.0, 73.6)	23.4% (0.0, 73.6)	0.0% (0.0, 100)	0.0% (0.0, 100)	0.0% (0.0, 100)

TABLE 5

Results from Estimation with Six Different Sets of Monthly Portfolio Returns

Reported are the RBIC estimates of the ranks of beta matrices from 6 different sets of U.S. stock portfolio returns. The RBICD estimates of the ranks of beta matrices are in parentheses. The individual rows of Table 5 report RBIC and RBICD estimation results and the adjusted R^2 from portfolio-by-portfolio time series regressions obtained using different sets of empirical factors (with the numbers of empirical factors used in parentheses in the first column). FF3, Carhart, FF5, HXZ, MOM, REV, CRR and JW, respectively, refer to the 3 Fama–French factors (FF3: VW, SMB and HML); the Carhart 4 factors (Carhart: FF3 and MOM); the five Fama–French factors (FF5: FF3, RMW and CMA); the Hou–Xue–Zhang 4 factors (HXZ: VW, SMB, IA and ROE); the momentum factor (MOM); the short-term and long-term reversal factors (REV); the five Chen–Roll–Ross macroeconomic factors (CRR: MP, UI, DEI, UTS and UPR) and the 3 Jagannathan and Wang factors (JW: VW, PREM and LAB). The sample period is from Jan. 1972 to Dec 2011 (T = 480) in Panel A. Panels B and C report the rank estimation results for the sample periods during Jan. 1972 to Dec 1991 (T = 240) and Jan. 1992 to Dec 2011 (T = 240), respectively.

	25 Size and BM		30 Industrial		25 Size and BM + 30 Industrial		25 Size and Momentum		25 Op. Prof. and Investment		32 Size, Op. Prof. and Investment	
Panel A. Whole Sample Period during Jan. 1972-Dec 2011 (T = 480)												
Empirical Factors (K)	$\bar{R}^2$	RBIC (RBICD)	$\bar{R}^2$	RBIC (RBICD)	$\bar{R}^2$	RBIC (RBICD)	$\bar{R}^2$	RBIC (RDBIC)	$\bar{R}^2$	RBIC (RBICD)	$\bar{R}^2$	RBIC (RBICD)
FF3 (3)	91.8%	3 (3)	60.7%	3 (3)	73.3%	3 (3)	82.4%	3 (2)	81.1%	3 (2)	87.6%	3 (3)
Carhart (4)	91.9%	3 (3)	61.4%	3 (3)	73.8%	4 (3)	91.4%	4 (3)	81.3%	3 (3)	87.7%	3 (3)
FF5 (5)	91.8%	4 (4)	61.9%	4 (4)	74.0%	4 (4)	82.5%	3 (2)	83.5%	4 (4)	89.6%	5 (5)
HXZ (4)	89.0%	4 (4)	60.6%	4 (4)	72.1%	4 (4)	84.3%	4 (3)	82.2%	3 (2)	88.2%	4 (4)
MOM and REV (3)	12.2%	1 (1)	7.9%	1 (1)	9.6%	2 (2)	22.2%	2 (2)	8.6%	1 (1)	11.1%	1 (1)
CRR (5)	1.3%	0 (0)	1.1%	0 (0)	1.2%	0 (0)	2.0%	0 (0)	0.7%	0 (0)	1.3%	0 (0)
JW (3)	73.6%	1 (1)	57.2%	1 (1)	63.8%	1 (1)	72.1%	1 (1)	78.1%	1 (1)	75.0%	1 (1)
All together (17)	92.3%	4 (4)	64.5%	5 (4)	75.7%	5 (4)	92.0%	4 (3)	84.2%	4 (4)	90.3%	5 (5)
FF5+ HXZ+MOM (8)	92.2%	4 (4)	63.5%	5 (5)	75.1%	5 (4)	91.8%	4 (3)	84.1%	4 (4)	90.2%	5 (5)

Panel B. Sample Period during Jan. 1972-Dec 1991 (T = 240)

FF3 (3)	94.1%	3 (3)	71.7%	3 (2)	81.1%	3 (3)	88.5%	2 (2)	85.8%	3 (2)	90.9%	3 (2)
Carhart (4)	94.2%	3 (3)	72.0%	3 (2)	81.3%	3 (3)	93.8%	3 (3)	85.9%	3 (2)	91.0%	3 (2)
FF5 (5)	93.9%	3 (3)	72.4%	4 (3)	81.4%	4 (4)	88.4%	2 (2)	87.6%	4 (3)	91.9%	4 (4)
HXZ (4)	92.6%	3 (3)	71.8%	3 (2)	80.5%	3 (3)	89.8%	3 (3)	86.4%	3 (2)	91.2%	3 (3)
MOM and REV (3)	15.0%	1 (1)	9.6%	1 (1)	11.9%	1 (1)	19.8%	1 (1)	10.8%	1 (1)	13.7%	1 (1)
CRR (5)	8.3%	0 (0)	6.6%	0 (0)	7.3%	0 (0)	8.6%	0 (0)	6.7%	0 (0)	7.5%	0 (0)
JW (3)	79.6%	1 (1)	68.8%	1 (1)	73.3%	1 (1)	77.6%	1 (1)	83.5%	1 (1)	80.4%	1 (1)
All together (17)	94.2%	3 (3)	74.2%	3 (2)	82.6%	4 (4)	94.0%	3 (3)	88.0%	4 (3)	92.4%	5 (4)
FF5 + HXZ + MOM (8)	94.0%	3 (3)	73.0%	4 (3)	81.8%	4 (4)	93.9%	3 (3)	87.8%	4 (3)	92.2%	4 (4)

Panel C. Sample Period during Jan. 1992-Dec 2011 (T = 240)

FF3 (3)	90.0%	3 (3)	54.8%	3 (3)	68.6%	3 (3)	78.5%	3 (3)	77.7%	3 (3)	85.8%	3 (3)
Carhart (4)	90.2%	3 (3)	56.2%	3 (3)	69.5%	4 (3)	90.3%	4 (3)	78.2%	4 (3)	86.0%	4 (3)
FF5 (5)	90.1%	4 (3)	55.4%	4 (3)	69.0%	4 (4)	78.2%	3 (3)	80.3%	3 (3)	87.9%	4 (3)
HXZ (4)	87.0%	4 (3)	53.5%	2 (2)	66.6%	3 (3)	81.1%	3 (2)	78.9%	3 (2)	86.1%	4 (3)
MOM and REV (3)	15.2%	2 (2)	11.5%	1 (1)	12.9%	2 (2)	30.3%	2 (2)	12.6%	2 (2)	14.5%	2 (2)
CRR (5)	0.5%	0 (0)	0.1%	0 (0)	0.3%	0 (0)	0.4%	0 (0)	0.3%	0 (0)	0.6%	0 (0)
JW (3)	67.3%	1 (1)	48.8%	1 (1)	56.0%	1 (1)	68.0%	1 (1)	73.0%	1 (1)	69.7%	1 (1)
All together (17)	91.2%	4 (4)	59.1%	4 (4)	71.7%	5 (4)	91.2%	4 (3)	81.8%	4 (3)	89.0%	5 (4)
FF5 + HXZ + MOM (8)	90.9%	4 (4)	57.7%	4 (4)	70.7%	5 (4)	90.8%	4 (3)	81.4%	4 (3)	88.8%	5 (4)

TABLE 6

Results from Estimation with Six Different Sets of Quarterly Portfolio Returns

Reported are the RBIC estimates of the ranks of beta matrices from 6 different sets of U.S. stock portfolio returns. The RBICD estimates of the ranks of beta matrices are in parentheses. The individual rows of Table 6 report the rank estimation results and the average adjusted R^2 from portfolio-by-portfolio time series regressions obtained using different sets of empirical factors (with the numbers of empirical factors used in parentheses in the first column). FF3, Carhart, FF5 and HXZ, respectively, refer to the 3 Fama–French factors (FF3: VW, SMB and HML); the Carhart 4 factors (Carhart: FF3 and MOM); the five Fama–French factors (FF5: FF3, RMW and CMA) and the Hou–Xue–Zhang 4 factors (HXZ: VW, SMB, IA and ROE). The additional models we consider and their empirical factors are the CCAPM with CG; the Lettau–Ludvigson (LL) model with CAY, CG and CAY×CG; the Yogo (Yogo) model with VW, DCG and NDCG; the Santos–Veronesi (SV) model with VW and VW×LC and the Li–Vassalou–Xing (LVX) model with DHH, DCORP and DHCORP. The sample period is from the first quarter of 1972 to the fourth quarter of 2011 (T = 160).

	25 Size and Book to Market		30 Industrial Portfolios		25 Size and BM + 30 Industrial		25 Size and Momentum		25 Op. Prof. and Investment		32 Size, Op. Prof. and Inv.	
Empirical Factors (K)	$\bar{R}^2$	RBIC (RBICD)	$\bar{R}^2$	RBIC (RBICD)	$\bar{R}^2$	RBIC (RBICD)	$\bar{R}^2$	RBIC (RBICD)	$\bar{R}^2$	RBIC (RBICD)	$\bar{R}^2$	RBIC (RBICD)
FF3 (3)	93.0%	3 (3)	63.7%	3 (3)	76.2%	3 (2)	85.5%	3 (3)	83.1%	3 (2)	89.2%	3 (2)
Carhart (4)	93.0%	3 (3)	64.0%	3 (3)	76.4%	3 (2)	92.3%	3 (3)	83.0%	4 (3)	89.3%	3 (2)
FF5 (5)	93.1%	4 (4)	65.2%	4 (4)	77.2%	4 (3)	85.3%	4 (4)	85.5%	3 (2)	91.4%	4 (3)
HXZ (4)	90.4%	3 (3)	64.6%	3 (3)	75.6%	3 (3)	87.5%	4 (3)	84.2%	3 (3)	90.0%	3 (2)
FF5+ HXZ+MOM (8)	93.6%	4 (3)	66.5%	4 (3)	78.1%	4 (3)	93.2%	5 (4)	85.8%	4 (4)	91.8%	4 (3)
CCAPM (1)	1.3%	0 (0)	0.9%	0 (0)	1.1%	0 (0)	1.2%	0 (0)	1.8%	0 (0)	1.7%	0 (0)
LL (3)	1.3%	0 (0)	1.2%	0 (0)	1.3%	0 (0)	1.4%	0 (0)	2.7%	0 (0)	2.0%	0 (0)
Yogo (3)	75.8%	1 (1)	59.8%	1 (1)	66.6%	1 (1)	74.0%	1 (1)	80.3%	1 (1)	77.6%	1 (1)
SV (2)	76.3%	1 (1)	60.9%	1 (1)	67.5%	2 (2)	74.7%	1 (1)	80.9%	1 (1)	78.0%	1 (1)
LVX (3)	7.0%	0 (0)	3.2%	0 (0)	4.8%	0 (0)	7.0%	0 (0)	2.9%	0 (0)	5.9%	0 (0)
All together (17)	93.8%	4 (3)	68.6%	4 (3)	79.6%	4 (3)	93.6%	5 (5)	86.1%	4 (3)	92.0%	3 (2)

TABLE 7

Results from Estimation with Different Sets of Monthly Individual Stock Returns

Reported are the average RBIC and RBICD estimates over 10,000 random samples of the ranks of beta matrices from individual U.S. stock returns. The standard deviations from the estimates are in parentheses. The individual rows of Table 7 report RBIC and RBICD estimation results and the adjusted R^2 from asset-by-asset time series regressions. We report in parenthesis the standard deviation of the estimators over 10,000 different samples. The results are obtained using different sets of empirical factors (with the numbers of empirical factors used in parentheses in the first column). FF3, Carhart, FF5, HXZ, MOM, REV, CRR and JW, respectively, refer to the 3 Fama–French factors (FF3: VW, SMB and HML); the Carhart 4 factors (Carhart: FF3 and MOM); the 5 Fama–French factors (FF5: FF3, RMW and CMA); the Hou–Xue–Zhang 4 factors (HXZ: VW, SMB, IA and ROE); the momentum factor (MOM); the short-term and long-term reversal factors (REV); the 5 Chen–Roll–Ross macroeconomic factors (CRR: MP, UI, DEI, UTS and UPR) and the 3 Jagannathan and Wang factors (JW: VW, PREM and LAB). Panels A and B report the results from 10,000 random samples of 50 and 200 stocks, respectively.

Panel A. Results from 10,000 Random Samples of 50 Stocks

Empirical Factors (*K*)	1972 – 2011 (T = 480, N = 50)			1972 – 1991 (T = 240, N = 50)			1992 – 2011 (T = 240, N = 50)		
	RBIC	RBICD	$\bar{R}^2$	RBIC	RBICD	$\bar{R}^2$	RBIC	RBICD	$\bar{R}^2$
FF3 (3)	2.78 (0.41)	2.21 (0.45)	0.28 (0.014)	2.13 (0.34)	1.90 (0.30)	0.31 (0.018)	2.42 (0.51)	2.25 (0.55)	0.20 (0.019)
Carhart (4)	2.79 (0.40)	2.50 (0.52)	0.29 (0.015)	2.16 (0.37)	1.89 (0.31)	0.32 (0.018)	2.44 (0.51)	2.32 (0.53)	0.21 (0.020)
FF5 (5)	2.93 (0.31)	2.24 (0.46)	0.29 (0.014)	2.55 (0.50)	1.93 (0.28)	0.32 (0.018)	2.45 (0.51)	2.23 (0.57)	0.20 (0.020)
HXZ (4)	2.02 (0.13)	1.84 (0.37)	0.27 (0.014)	2.00 (0.07)	1.67 (0.47)	0.31 (0.018)	1.98 (0.26)	1.65 (0.48)	0.19 (0.018)
MOM and REV (3)	0.33 (0.47)	0.32 (0.46)	0.06 (0.006)	0.01 (0.11)	0.01 (0.11)	0.06 (0.006)	0.48 (0.50)	0.34 (0.47)	0.06 (0.009)
CRR (5)	0.00 (0.00)	0.00 (0.00)	0.01 (0.002)	0.00 (0.00)	0.00 (0.00)	0.03 (0.005)	0.00 (0.00)	0.00 (0.00)	0.01 (0.003)
JW (3)	1.00 (0.00)	1.00 (0.00)	0.24 (0.013)	1.00 (0.00)	0.95 (0.23)	0.26 (0.018)	1.00 (0.00)	0.99 (0.06)	0.14 (0.016)
All together (17)	2.95 (0.38)	2.41 (0.54)	0.31 (0.016)	2.66 (0.48)	1.91 (0.32)	0.33 (0.019)	2.48 (0.52)	2.30 (0.54)	0.23 (0.022)

Panel B. Results from 10,000 Random Samples of 200 Stocks

Empirical Factors (*K*)	1972 – 2011 (*T* = 480, *N* = 200)			1972 – 1991 (*T* = 240, *N* = 200)			1992 – 2011 (*T* = 240, *N* = 200)		
	RBIC	RBICD	$\bar{R}^2$	RBIC	RBICD	$\bar{R}^2$	RBIC	RBICD	$\bar{R}^2$
FF3 (3)	2.94 (0.23)	2.38 (0.48)	0.28 (0.004)	3.00 (0.00)	3.00 (0.01)	0.31 (0.008)	3.00 (0.00)	3.00 (0.01)	0.19 (0.009)
Carhart (4)	2.99 (0.08)	2.99 (0.11)	0.29 (0.004)	3.98 (0.13)	3.96 (0.20)	0.32 (0.008)	4.00 (0.01)	3.99 (0.05)	0.20 (0.009)
FF5 (5)	2.98 (0.14)	2.45 (0.50)	0.29 (0.004)	4.90 (0.30)	4.86 (0.34)	0.32 (0.008)	4.93 (0.25)	4.89 (0.31)	0.20 (0.009)
HXZ (4)	2.00 (0.01)	2.00 (0.01)	0.27 (0.004)	3.92 (0.26)	3.89 (0.30)	0.31 (0.008)	3.99 (0.04)	3.99 (0.07)	0.19 (0.009)
MOM and REV (3)	0.20 (0.39)	0.18 (0.38)	0.06 (0.003)	2.65 (0.49)	2.58 (0.51)	0.06 (0.003)	2.98 (0.12)	2.97 (0.16)	0.06 (0.004)
CRR (5)	0.00 (0.00)	0.00 (0.00)	0.01 (0.000)	3.17 (0.58)	2.99 (0.58)	0.03 (0.002)	3.32 (0.61)	3.17 (0.61)	0.01 (0.001)
JW (3)	1.00 (0.00)	1.00 (0.00)	0.24 (0.004)	2.07 (0.52)	1.98 (0.51)	0.26 (0.008)	2.27 (0.52)	2.20 (0.51)	0.14 (0.007)
All together (17)	3.60 (0.49)	3.54 (0.50)	0.31 (0.005)	11.84 (0.70)	11.55 (0.71)	0.33 (0.008)	12.49 (0.70)	12.23 (0.71)	0.23 (0.010)